Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 5, 1998

FORCED NONLINEAR OSCILLATIONS OF A GAS BUBBLE IN A LARGE SPHERICAL FLASK (RESONATOR) FILLED WITH A FLUID

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Radial oscillations of a gas bubble in a large spherical flask filled with a fluid are considered. We derive an equation of the change of the bubble radius by the known law of pressure variation at the boundary of the liquid volume (the law of motion of the piston) for a period of time during which, repeatedly reflected from the piston, the leading front of the reflected-from-the bubble perturbations reaches the bubble. For further calculations of the change of the bubble radius, recurrent relations which include the wave reflected from the bubble in the previous cycle and its subsequent reflection from the piston are obtained. Under harmonic action of the piston on the fluid–bubble system, a certain periodic regime with a package of bubble oscillations is established.

Introduction. Short pulses of high temperature and pressure in the gas phase can be reached by vibrational or acoustic action on the fluid–bubble system in a certain vessel. In the resonance, almost adiabatic supercompression of the bubble, leading to periodic supershort ($\approx 10^{-11}$ sec) and superhigh temperatures of the gas ($\approx 10^6$ K) in its center, even bubble luminescence can be observed. This phenomenon is called sonoluminescence [1–3]. The possibility of attaining the higher maximum temperatures was considered in [4, 5]. Here the basic physical mechanism is the radially convergent, inertial motion of the fluid, permitting a significant fraction of the mechanical energy supplied to be concentrated in the bubble in the form of gas-phase internal energy. The Rayleigh–Lamb equation, obtained within the framework of the incompressible fluid, or its modified forms making allowance for, e.g., acoustic radiation, is usually used to describe the dynamics of symmetrical oscillations of the bubbles [6]:

$$a\ddot{a} + \frac{3}{2}\dot{a}^2 + 4\,\frac{\nu_l \dot{a}}{a} = \frac{p_g - p_{l\infty}}{\rho_l}.$$

Here a is the bubble radius, p_g and $p_{l\infty}$ is the pressure in the bubble and in the fluid far from the bubble, ρ is the density of the fluid, and ν_l is the kinematic viscosity of the fluid. This usage assumes that the characteristic time of propagation of a sound wave through the liquid volume is $t_c = R/C_l$ (R is the characteristic linear dimension of the volume and C_l is the velocity of sound in the fluid) is much less than the period t_M of eigenoscillations of the bubbles ($t_c \ll t_M$), determined by the Minaert formula. However, when the radial motion of the bubble is considered in a "large" volume ($t_c \geqslant t_M$), the description of the bubble dynamics by means of the Rayleigh–Lamb equations with the adopted value $p_{l\infty}$ of the pressure at the boundary of the liquid volume is not justified. The greatest effect in the attainment of superhigh temperatures in gas bubbles can be obtained precisely in "large" volumes. This is primarily connected with the fact that if in the case of action of the pressure on a "small" volume, energy accumulation mainly occurs in the form of the kinetic energy of the radial fluid motion, then in the case of a "large" volume, first, energy accumulation is basically connected with fluid compressibility and, second, the wave-assisted transfer of this energy to the volume's

Institute of Mechanics, Ural Scientific Center, Russian Academy of Sciences, Ufa 450000. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 5, pp. 77–87, September–October, 1998. Original article submitted December 31, 1996; revision submitted December 16, 1996.

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center and its concentration (cumulation) occur. Here the most energetically favorable regime is realized at frequencies equal to those of free eigenoscillations of the spherical liquid volume.

We note that the Herring–Flynn equation and the refinement proposed in [7] are not suitable for calculations of forced gas-bubble oscillations in a "large" but finite volume. The Herring–Flynn equation describes only the initial stage of dynamics, when the perturbations reflected from the external boundary of the liquid volume did not yet return to the bubble. Moreover, the procedure of deriving an improved equation was not correct, because in fact the equal-to-infinity term was ignored. We note that the physical interpretation of the presence of this term is very simple and is connected with the fact that any convergent-from-infinity spherically symmetrical pressure perturbation has an infinite amplitude at a finite distance.

1. Basic Equations. We shall consider radial oscillations of a gas bubble in a compressible fluid. The equations of fluid motion about the bubble are taken in the form

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial r}\frac{w^2}{2}=-\frac{1}{\rho_l^0}\frac{\partial p_l}{\partial r},\quad \frac{\partial \rho_l}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho_l w r^2)=0,\quad p_l=p_0+C_l^2(\rho_l-\rho_{l0}),\tag{1.1}$$

where w is the fluid velocity. Here and below, the subscript l corresponds to the fluid parameters, and the subscript 0 refers to the undisturbed state of the system. The fluid compressibility is adopted in a linear approximation. We assume that the pressure and the temperature in the gas phase are uniform. We also assume that the gas behavior is polytropic and, in particular, adiabatic. Then its pressure p_g is uniquely determined via the current value of the bubble radius:

$$p_g=p_0(a_0/a)^{3\gamma},\tag{1.2}$$

where γ is the polytropic exponent of the gas.

For weak perturbations in the fluid ($w \ll C_l$), from Eq. (1.1) it follows the wave equation for pressure perturbations far from the bubble ($r \gg a$)

$$\frac{\partial^2\Delta p_l}{\partial t^2}=C_l^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Delta p_l}{\partial r}\right)\quad (\Delta p_l=p_l-p_0).\tag{1.3}$$

For the velocity distribution, we have

$$w=-\frac{1}{\rho_{l0}}\int\limits_{-\infty}^{t}\frac{\partial\Delta p_l}{\partial r}\,dt.\tag{1.4}$$

In the general case, the solution of Eq. (1.3) is of the form $\Delta p_l=\Delta p_{\rm div}+\Delta p_{\rm con}$, where $\Delta p_{\rm con}=f_{\rm con}(t+r/C_l)/r$, $\Delta p_{\rm div}=f_{\rm div}(t-r/C_l)/r$ and $\Delta p_{\rm con}$ and $\Delta p_{\rm div}$ correspond to the convergent and divergent waves, respectively. Based on (1.4), for the velocity we can write

$$w=w_{\rm con}+w_{\rm div},$$
$$w_{\rm div}=\frac{\Delta p_{\rm div}}{\rho_l C_l}+\frac{1}{\rho_{l0}r}\int\limits_{-\infty}^{t}\Delta p_{\rm div}dt,\qquad w_{\rm con}=-\frac{\Delta p_{\rm con}}{\rho_l C_l}+\frac{1}{\rho_{l0}r}\int\limits_{-\infty}^{t}\Delta p_{\rm con}dt.\tag{1.5}$$

2. Initial Stage of Oscillations. Let a piston in equilibrium ($w=0$ and $p_l=p_g=p_0$) begin to act at $t=0$ on the fluid–gas bubble system through the boundary of a spherical liquid volume of radius R. We present the law of pressure variation behind the piston as follows:

$$p^{(R)}(t)=p_0+\Delta p^{(R)}(t).\tag{2.1}$$

The velocity of the piston and the pressure perturbation behind it are related by the relation $w^{(R)}(t)=-\Delta p^{(R)}(t)/\rho_{l0}C_l$. Then a pressure wave is expected to propagate over the initially quiescent fluid:

$$\Delta p_{\rm con}=\frac{R}{r}\Delta p^{(R)}\Big(t+\frac{r-R}{C_l}\Big).\tag{2.2}$$

Having reached the leading wavefront moving by the law $r_w=R-C_lt$, this wave reflects from the bubble surface ($r_w = a_0$). We search for pressure perturbations that correspond to the reflected wave in the form

$$\Delta p_{\rm div} = \frac{R}{r}\,\Delta p^{(R)}_{\rm div}\Big(t - \frac{r}{C_l}\Big). \tag{2.3}$$

For $t \geqslant t_c$ [$t_c = (R - a_0)/C_l \approx R/C_l$], from the pressure-continuity condition on the bubble surface ($r = a$) we write

$$(\Delta p_{\rm con} + \Delta p_{\rm div})\Big|_{r=a} = p_g(t) - p_0. \tag{2.4}$$

Substituting expressions (2.2) and (2.3) into (2.4), we obtain

$$\frac{R}{a}\Big(\Delta p^{(R)}\Big(t + \frac{a-R}{C_l}\Big) + \Delta p^{(R)}_{\rm div}\Big(t - \frac{a}{C_l}\Big)\Big) = p_g(t) - p_0.$$

For the wave reflected from the gas cavity, we have

$$\Delta p_{\rm div} = -\frac{R}{r}\,\Delta p^{(R)}\Big(t + \frac{2a - r - R}{C_l}\Big) + \frac{a}{r}\Big(p_g\Big(t + \frac{a-r}{C_l}\Big) - p_0\Big). \tag{2.5}$$

In addition, for $t \geqslant t_c$ the sum of the velocity perturbations which correspond to the incident and reflected waves should satisfy the condition

$$(w_{\rm con} + w_{\rm div})\Big|_{r=a} = \frac{da}{dt}. \tag{2.6}$$

Based on expressions (1.5), with allowance for (2.2) and (2.5), for the velocity field we obtain

$$\begin{gathered} w = w_{\rm con} + w_{\rm div}, \quad w_{\rm con} = -\frac{R}{\rho_{l0}C_l r}\,p^{(R)}\Big(t + \frac{r-R}{C_l}\Big) + \frac{R}{\rho_{l0}r^2}\int\limits_0^t \Delta p^{(R)}\Big(t + \frac{r-R}{C_l}\Big)\,dt, \\ w_{\rm div} = -\frac{R}{\rho_{l0}C_l r}\,\Delta p^{(R)}\Big(t + \frac{2a-r-R}{C_l}\Big) + \frac{a}{\rho_{l0}C_l r}\Big(p_g\Big(t + \frac{a-r}{C_l}\Big) - p_0\Big) \\ -\frac{R}{\rho_{l0}r^2}\int\limits_0^t \Delta p^{(R)}\Big(t + \frac{2a-r-R}{C_l}\Big)\,dt + \frac{1}{\rho_{l0}r^2}\int\limits_0^t a\Big(t + \frac{a-r}{C_l}\Big)\Big(p_g\Big(t + \frac{a-r}{C_l}\Big) - p_0\Big)\,dt \quad (t \geqslant t_c). \end{gathered} \tag{2.7}$$

We note that the solution (2.7) describes the velocity distribution in the liquid volume $a(t) \leqslant r \leqslant R$ for the period of time $t_c \leqslant t \leqslant 2t_c$. Substituting (2.7) into (2.6), we obtain the integrodifferential equation

$$-2\,\frac{R}{\rho_{l0}C_l a(t)}\,\Delta p^{(R)}\Big(t + \frac{a(t)-R}{C_l}\Big) + \frac{1}{\rho_{l0}C_l}(p_g(t) - p_0) + \frac{1}{\rho_{l0}a^2(t)}\int\limits_0^t a(t)(p_g(t) - p_0)\,dt = \frac{da}{dt}.$$

Ignoring $a(t)$ compared with R, after some transformations we obtain

$$\rho_{l0}\,\frac{d}{dt}\,(a^2\dot a) = a(p_g(t) - p_0) - \frac{d}{dt}\Big\{\frac{a^2}{C_l}\Big[\frac{2R}{a}\,\Delta p^{(R)}(t - t_c) - (p_g(t) - p_0)\Big]\Big\}. \tag{2.8}$$

According to the known law of pressure variation $\Delta p^{(R)}(t)$ at the boundary of a liquid volume [or the law of piston motion $w^{(R)}(t)$], Eq. (2.8) allows one to determine the change in the gas-bubble radius for a period of time during which, repeatedly reflected from the piston ($r = R$), the leading front of the perturbations, reflected from it, reaches the bubble ($t_c \leqslant t \leqslant 3t_c$).

In the case of constant pressure in the bubble ($p_g = p_0$, which can occur for vapor bubbles), from Eq. (2.8) we obtain

$$a\dot a = -\frac{2R}{\rho_{l0}C_l}\,\Delta p^{(R)}(t - t_c). \tag{2.9}$$

Here the equilibrium position of the bubble ($\dot a = 0$) for $t = t_c$ is taken into account. Based on (2.9), we obtain

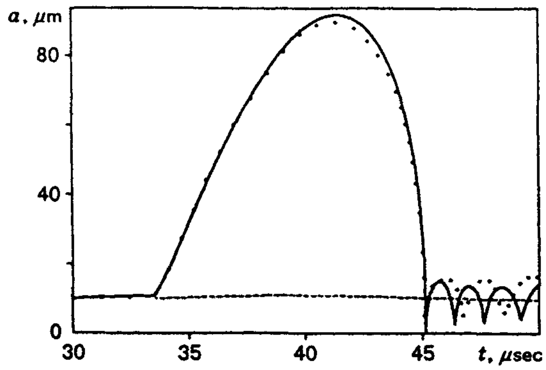

Fig. 1

the solution

$$a^2 - a_0^2 = -\frac{4R}{\rho_{l0} C_l} \int_{t_c}^{t} \Delta p^{(R)}(t - t_c)\, dt. \tag{2.10}$$

For the harmonic law of pressure variation on the piston, we have

$$\Delta p^{(R)} = \Delta p_A^{(R)} \sin(\omega t). \tag{2.11}$$

It follows from (2.10) that

$$a = \sqrt{a_0^2 - \frac{8R}{\rho_{l0} C_l \omega} \Delta p_A^{(R)} \sin^2\left(\frac{\omega(t - t_c)}{2}\right)} \qquad (t > t_c). \tag{2.12}$$

Using solutions (2.7) for the velocity distribution or (2.2) and (2.3) for the pressure distribution (based on the impulse-momentum equations), we obtain the acceleration field. Assuming that on the bubble surface, the condition

$$\left(\frac{\partial w_{\text{con}}}{\partial t} + \frac{\partial w_{\text{div}}}{\partial t}\right)\bigg|_{r=a} = \frac{d^2 a}{dt^2}$$

is satisfied, we have

$$\rho_{l0} a \frac{d^2 a}{dt^2} = p_g(t) - p_0 - \frac{d}{dt}\left\{\frac{2R}{C_l} \Delta p^{(R)}(t - t_c) - \frac{a}{C_l}(p_g(t) - p_0)\right\}. \tag{2.13}$$

Equation (2.13) differs from (2.8). It is associated with the fact that the derivation of these nonlinear equations with the use of the solution of the linear problem was not, generally speaking, correct. In the first case, the velocities were obtained with a variable value of the bubble radius [condition (2.6)] and, in the second case, the common point is acceleration. In linearized form, the two equations coincide.

It is noteworthy that Eqs. (2.8) or (2.13), obtained from the linear wave equations, can be used for fairly weak actions on the fluid–bubble system under which the rate of change of the bubble radius is significantly less compared to the velocity of sound in the fluid ($\dot{a} \ll C_l$) and the displacements of the fluid δr are small ($\delta r \ll r$ and, in particular, $\delta a \ll a$). A comparison of Eq. (2.13) with the Rayleigh–Lamb equation shows that the term $(3/2)\rho_{l0}\dot{a}^2$ "is lost" on the right-hand side. Supplementing the equation by this term and the viscosity-induced term, we obtain

$$\rho_{l0}\left(a\ddot{a} + \frac{3}{2}\dot{a}^2 + 4\frac{\nu_l \dot{a}}{a}\right) = p_g - p_0 - \frac{d}{dt}\left\{\frac{a}{C_l}\left[\frac{2R}{a}\Delta p^{(R)}(t - t_c) - (p_g - p_0)\right]\right\}. \tag{2.14}$$

Apparently, the range of applicability of Eq. (2.14) is much wider than that of Eq. (2.13). It is supported by comparison of the solution of Eq. (2.14) (the solid curve in Fig. 1) with the numerical solution [5] (the dotted curve), obtained after integration of the complete system (1.1) for the following parameters of the water–air bubble system: $a_0 = 10^{-5}$ m, $R = 5 \cdot 10^{-2}$ m, $C_l = 1500$ m/sec, $\gamma = 1.4$, and $\nu_l = 0$. The law of pressure variation at the boundary of the liquid volume ($r = R$) for $t > 0$ is taken in the form (2.11). The following values for the amplitude of pressure and circular frequency are used: $\Delta p_A^{(R)} = 0.25 \cdot 10^5$ Pa and $\omega = 2\pi \cdot 45$ kHz. The dashed curve was calculated by the Rayleigh–Lamb equation for the same parameters. We note that Moss et al. [5] solved the complete gas-dynamic problem with allowance for dissociation and ionization in the gas phase. The surprising proximity of the solid and dotted curves, particularly at the stage of the first slow expansion and compression of the bubble in an "extreme" regime, shows the advantage of Eq. (2.14), because its use instead of the complete system of hydrodynamic equations in a fluid (1.1) basically facilitates the procedure of numerical analysis of the dynamics of the fluid–bubble system. This circumstance is particularly important for calculations in a cyclical regime of action. The process of taking into account should include two stages. In the first stage, based on Eq. (2.14) (or its modifications with allowance for repeated reflections of the waves from the liquid boundary) the motion in the slow stage, in which energy accumulation occurs in the system, is determined. The gas behavior in the bubble at this stage can be described by the idealized pattern (1.2). Therefore, one needs to calculate the processes occurring in the fluid and in the gas in a lengthy formulation only for the fast stage of compression and expansion of the gas cavity, using the results of the first stage of calculation as the initial data. It is necessary to note that studying the processes near the moment of maximum compression of the bubble with allowance for the waves converging to the center and reflected from the center and the bubble walls is decisive for analysis of superhigh temperatures and pressures in the gas phase. If one considers the behavior of a gas bubble from a macroscopic point of view, when the variation of the radius, pressure, and temperature of the bubble as functions of time is of interest, it is possible to confine oneself to an approximate description of the gas behavior in the bubble. This is confirmed by Fig. 1, where the solid and dotted curves are in good agreement with each other after the first and subsequent "jumps off." Consequently, the use of the Rayleigh-Lamb equation for calculation of the bubble-radius oscillations is not justified.

The validity of Eq. (2.14) can be justified in another way. Under the wave action on the fluid–bubble system in the neighborhood of the gas bubble at a distance l, which is significantly smaller than the wavelength λ ($l \ll \lambda$), the compressibility effects are insignificant. We separate a certain effective volume of an incompressible fluid with the external radius a_{eff} ($\lambda \gg a_{\text{eff}} \gg a$) near the bubble. We assume that the fluid motion occurs according to the equation of a viscous incompressible fluid in this volume ($a_{\text{eff}} \geqslant r \geqslant a$) and the linear equations (1.3) and (1.4) outside this volume ($a_{\text{eff}} \leqslant r \leqslant R$). After transformations similar to those used in deriving (2.8) and assuming the satisfaction of the conditions $a_0 \ll a_{\text{eff}} \ll R$, we obtain Eq. (2.14). In other words, this equation follows from combining the solution of the wave equation with that of the equation of an viscous incompressible fluid. These aspects of the description of the bubble dynamics were analyzed in [4].

Since Eq. (2.13) describes the bubble dynamics prior to the moment $t = 3t_c$, it is necessary to refine this moment for further calculations ($t > 3t_c$) with allowance for the wave reflected from the gas bubble in the previous cycle and its subsequent reflection from the piston. One can adopt two patterns to describe the reflection of waves from the piston. In the first pattern, the pressure perturbation on the piston $\Delta p^{(R)}(t)$ is always a known specified function, and, in the second pattern, the law of piston motion $\Delta R(t)$ is specified relative to the initial position. In accordance with these patterns, the boundary conditions on the piston can be written as follows:

$$(p_{\text{con}} + p_{\text{div}})\Big|_{r=R} = \Delta p^{(R)}(t) \tag{2.15}$$

or

$$(w_{\text{con}} + w_{\text{div}})\Big|_{r=R} = w^{(R)}(t), \quad w^{(R)}(t) = \frac{dR(t)}{dt} \quad (t \geqslant 2t_c). \tag{2.16}$$

With allowance for the relations on the piston surface $w_{\text{con}} = -\Delta p_{\text{con}}/(\rho_{l0} C_l)$ and $w_{\text{div}} = \Delta p_{\text{div}}/(\rho_{l0} C_l)$ for $r = R$, the boundary condition (2.16) can be reduced to the form $(-\Delta p_{\text{con}} + \Delta p_{\text{div}})\Big|_{r=R} = \rho_{l0} C_l w^{(R)}(t)$. Using expression (2.5) and the boundary conditions (2.15) and (2.16), we obtain

$$\Delta p_{\rm con}\Big|_{r=R} = \Delta p^{(R)}_{(1)}(t) = \Delta p^{(R)}(t) + \Delta p^{(R)}\Big(t + \frac{2(a-R)}{C_l}\Big) - \frac{a}{R}\Big(p_g\Big(t + \frac{a-R}{C_l}\Big) - p_0\Big)$$

or

$$\Delta p_{\rm con}\Big|_{r=R} = \Delta p^{(R)}_{(1)}(t) = -\rho_{l0}C_l w^{(R)}(t) - \Delta p^{(R)}\Big(t + \frac{2(a-R)}{C_l}\Big) + \frac{a}{R}\Big(p_g\Big(t + \frac{a-R}{C_l}\Big) - p_0\Big).$$

Ignoring the term a compared with R and using (2.1), we obtain

$$\Delta p^{(R)}_{(1)}(t) = \Delta p^{(R)}(t) + \Delta p^{(R)}(t - 2t_c) - \frac{a(t-t_c)}{R}(p_g(t-t_c) - p_0)$$

or

$$\Delta p^{(R)}_{(1)}(t) = -\rho_{l0}C_l(w^{(R)}(t) - w^{(R)}(t-2t_c)) + \frac{a(t-t_c)}{R}(p_g(t-t_c) - p_0).$$

Continuing this consideration, for the nth cycle, where $(2n+1)t_c \leqslant t \leqslant (2n+3)t_c$ with $n > 1$, we obtain

$$\Delta p^{(R)}_{(n)}(t) = \Delta p^{(R)}(t) + \Delta p^{(R)}(t-2t_c) + \ldots + \Delta p^{(R)}(t-2nt_c) - \frac{a(t-t_c)}{R}(p_g(t-t_c) - p_0)$$
$$- \frac{a(t-3t_c)}{R}(p_g(t-3t_c) - p_0) - \ldots - \frac{a(t-(2n-1)t_c)}{R}(p_g(t-(2n-1)t_c) - p_0) \qquad (2.17)$$

or

$$\Delta p^{(R)}_{(n)}(t) = -\rho_{l0}C_l(w^{(R)}(t) - w^{(R)}(t-2t_c) + \ldots + (-1)^n w^{(R)}(t-2nt_c)) + \frac{a(t-t_c)}{R}(p_g(t-t_c) - p_0)$$
$$- \frac{a(t-3t_c)}{R}(p_g(t-3t_c) - p_0) + \ldots + (-1)^n\frac{a(t-(2n-1)t_c)}{R}(p_g(t-(2n-1)t_c) - p_0). \qquad (2.18)$$

It is easy to see that relations (2.17) and (2.18) can be written as the following recurrent relations:

$$\Delta p^{(R)}_{(n)}(t) = \Delta p^{(R)}(t) + \chi\Big(\Delta p^{(R)}_{(n-1)}(t-2t_c) - \frac{a(t-t_c)}{R}(p_g(t-t_c) - p_0)\Big), \quad \Delta p^{(R)}_{(0)}(t) = \Delta p^{(R)}(t)$$

or

$$\Delta p^{(R)}_{(n)}(t) = -\rho_l C_l w^{(R)}(t) - \chi\Big(\Delta p^{(R)}_{(n-1)}(t-2t_c) - \frac{a(t-t_c)}{R}(p_g(t-t_c) - p_0)\Big),$$
$$\Delta p^{(R)}_{(0)} = -\rho_l C_l w^{(R)}(t) \qquad (n \geqslant 1).$$

Here the coefficient χ was introduced to include the incomplete reflection on the piston.

If the pressure on the piston is given, its velocity is found from the expression

$$w^{(R)}(t) = \frac{\Delta p^{(R)}(t) - (1+\chi)\Delta p^{(R)}_{(n)}(t)}{\chi\rho_{l0}C_l}.$$

When the velocity of the piston is known, for the pressure on the piston we have

$$\Delta p^{(R)}(t) = \frac{(1+\chi)\Delta p^{(R)}_{(n)}(t) + \rho_{l0}C_l w^{(R)}(t)}{\chi}.$$

The law of radius variation from the moment $(2n+1)t_c$ to $(2n+3)t_c$ is determined by Eqs. (2.12) with a replacement of $\Delta p^{(R)}(t)$ by $\Delta p^{(R)}_{(n)}(t)$ from (2.17) or (2.18).

Figure 2 shows calculation results obtained by means of Eq. (2.14). The law of pressure variation on the piston is adopted in the form (2.11); here $\Delta p_A^{(R)} = 0.075\cdot 10^5$ Pa, $\omega^{(R)} = 2\pi\cdot 15$ kHz, and $\nu_l = 10^{-5}$ m^2/sec. It is noteworthy that the adopted value of the frequency of forced oscillations corresponds to the first resonance frequency of the spherical water volume of radius $R = 5\cdot 10^{-2}$ m. The gradual "swinging" of the bubble

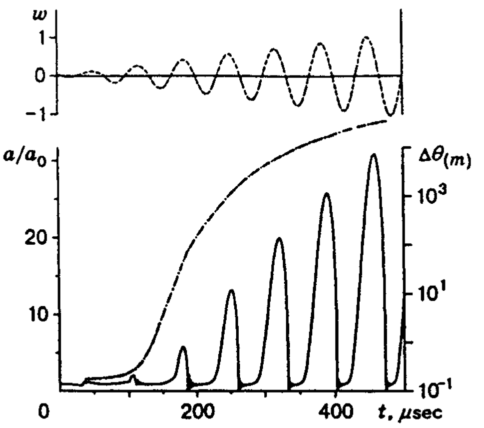

Fig. 2

radius (the solid curve) is traced on the diagram. If oscillations occur in a smoother regime in the initial stage, this regime gradually (beginning with the second cycle for our example) enters into a stage with a packet of oscillations, which consists of one leading oscillation with the more distinct compression and high-frequency "rattling." In this regime, for a prescribed pressure variation on the piston, the increase in the oscillation frequency of the bubble radius occurs until the leading-oscillation period $t^{(m)}$ reaches the phase of forced oscillations $[t^{(m)} < t^{(f)}]$. At this moment, the oscillations of bubbles "detune." The subsequent oscillations become irregular, and the oscillation frequency of the bubble radius stops to increase. As the effective parameter, which determines the gas state at the moment of maximum compression of the bubble, one can use the radial velocity of the bubble w_i in the compression stage, when the bubble radius is equal to the initial radius $[a(t_i) = a_0]$. Equating the kinetic energy of the radial motion at this velocity, calculated as for the incompressible fluid, to the increment of the bubble's internal energy as the radius changes from a_0 to the value at the moment of maximum compression a_m [4], we obtain an expression for the maximum possible variation in the temperature of a perfect gas:

$$\Delta\theta_{(m)} = \frac{\Delta T_{(m)}}{T_0} = \frac{3}{2}(\gamma - 1)\frac{\rho_{l0}}{p_0}w_{(i)}^2 \qquad (\Delta T_{(m)} = T_{(m)} - T_0).$$

The dependence of $\Delta\theta_{(m)}$ on time is shown in Fig. 2 (the dot-and-dashed curve). For the stage with a packet of oscillations, the value of the parameter $\Delta\theta_{(m)}$ was obtained for the leading oscillation. The parameter $\Delta\theta_{(m)}$ determines the energy "pumped" into the bubble. This parameter is directly related to the maximum possible temperature of a dense nonperfect gas in the strong compression of the bubble, but, generally speaking, it is not equal to this temperature. The dashed curve corresponds to the dimensionless velocity of the piston $W^{(R)} = -\rho_{l0}C_l w^{(R)}/p_0$ (this form of the dimensionless velocity is convenient for comparison with the value of the pressure behind the piston, which is induced by the piston motion according to the known law). This curve was drawn, ignoring the anomalously large peaks connected with high-frequency "rattling" of the bubbles. As some estimates show, in real systems these peaks of pressure because of the fluid viscosity and nonlinear effects indeed damp as the waves propagate from the bubble to the piston. The diagram shows that the gradual "swinging" of the velocity, and hence of the oscillation amplitude, of the piston occurs.

Figure 3 illustrates the dynamics of bubble oscillation with variation of the piston velocity according to the law

$$w^{(R)} = w_A^{(R)}\sin(\omega^{(R)}t). \tag{2.19}$$

The following values of the amplitude of the velocity oscillation and of the reflection coefficient on the piston were used: $w_A^{(R)} = 0.017$ m/sec, $\Delta p_A^{(R)} = 0.25$ Pa $[\Delta p_A^{(R)} = \rho_{l0}C_l w_A^{(R)}]$, and $\chi = 0.5$; the other parameters are the same as in the previous examples. It follows from the curves that in the case where the velocity of the piston is given in the form (2.19), the bubble oscillations reach a certain periodic regime. An analysis

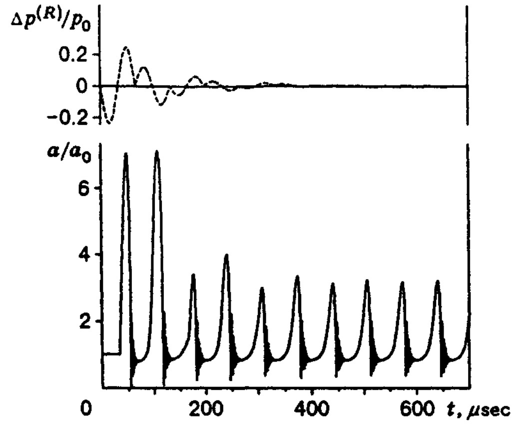

Fig. 3

of the calculation results as applied to experimental data [2] allows one to conclude that the oscillations in the initial stage correspond, to a large extent, to the oscillations in the regime of a piston at a specified pressure. In the initial stage, the "strength" of the experimental setup is determined by its possibility to excite pressure oscillations of maximum amplitude. As the amplitude of forced oscillations of the bubble increases, the system gradually reaches a steady-state periodic regime, in which the oscillation parameters are limited by the amplitude of displacements of the piston (the flask walls) $\Delta R_A = w_A^{(R)}/\omega^{(R)}$. The dashed curve in Fig. 3 corresponds to the dimensionless pressure perturbation on the piston $\Delta p^{(R)}/p_0$ without allowance for high-frequency peaks in the pressure. It is seen that, as the regime of steady-state oscillations is reached, the pressure amplitude on the piston, which corresponds to the oscillation of the piston by the law (3.1), damps and only the pressure peaks associated with bubble oscillations remain. We note that in the case of the absence of a bubble, in a steady-state regime the pressure at the piston in the center of the liquid volume should be equal to zero, because the frequency of oscillations $\omega^{(R)}$ is one of the eigenfrequencies of oscillations of the liquid volume of radius R.

3. Steady-State Forced Oscillations. We shall consider the motion of the fluid–bubble system under the harmonic action of the piston on this system. For definiteness, we assume that the piston executes simple harmonic motions with the circular frequency

$$\omega^{(R)}(t^{(R)} = 2\pi/\omega^{(R)}), \qquad \Delta R = \Delta R_A \sin(\omega^{(R)} t). \tag{3.1}$$

It follows from the previous analysis that precisely this formulation for the piston is most adequate as applied to the experimental data of [1], when a certain periodic regime is established. In addition, we assume that the frequency of oscillations of the piston coincides with one of the eigenfrequencies of the liquid volume of radius R for free oscillations [$\omega^{(R)} = \omega_c k$, $\omega_c = \pi C_l/R^{(R)}$, and k is an integer].

As has already been mentioned, under these conditions of the action on the fluid–gas bubble system a certain periodic regime with a packet of bubble oscillations is established:

$$a(t + t^{(R)}) = a(t), \qquad p_g(t + t^{(R)}) = p_g(t), \qquad p_{\rm con}(t + t^{(R)}) = p_{\rm con}(t),$$
$$p_{\rm div}(t + t^{(R)}) = p_{\rm div}(t), \qquad \Delta p^{(R)} = p_{\rm con}^{(R)} + p_{\rm div}^{(R)} \qquad \left(\Delta p^{(R)} = \Delta p\big|_{r=R}\right).$$

Here $p_{\rm div}^{(R)}$ and $p_{\rm con}^{(R)}$ are the components of the pressure perturbations on the piston surface, which correspond to the divergent (from the center) and convergent (to the center) waves.

As the boundary condition on the piston ($r = R$), we use a condition similar to (2.15):

$$-\Delta p_{\rm con}^{(R)} + \chi p_{\rm div}^{(R)} = \rho_{l0} C_l w^{(R)}(t) \qquad \left(w^{(R)}(t) = \frac{dR(t)}{dt}\right).$$

Here χ is the reflection coefficient on the piston ($\chi = 1$ and $\chi = 0$ correspond to the total reflection and the

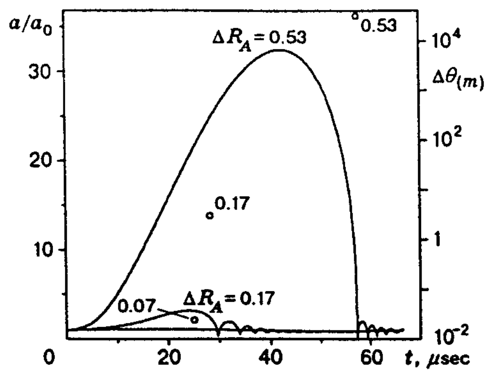

Fig. 4

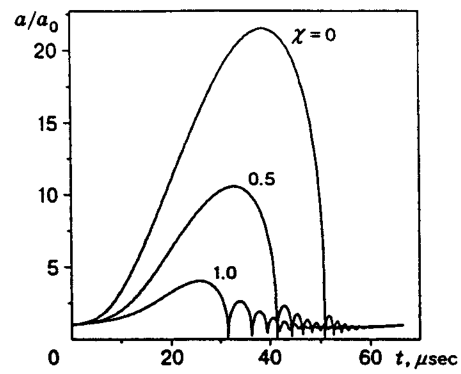

Fig. 5

absence of reflection on the piston).

From the pressure-continuity condition, at the boundary of the gas bubble, with allowance for $a \ll R$ we write

$$\Delta p_{\rm div}^{(R)}(t - t_c) + \Delta p_{\rm con}^{(R)}(t + t_c) = \frac{a(t)}{R}\,(p_g(t) - p_0). \tag{3.2}$$

Taking into account that the quantity $2t_c$ is the period of the functions $\Delta p_{\rm con}^{(R)}(t)$ and $\Delta p_{\rm div}^{(R)}(t)$, from (3.2) we obtain $\Delta p^{(R)}(t - t_c) = (a(t)/R)\,(p_g(t) - p_0)$. After appropriate transformations and the arguments as in deriving (2.14), from condition (2.6) we have

$$\rho_{l0}\Big(a\ddot{a} + \frac{3}{2}\,\dot{a}^2 + 4\,\frac{\nu_l \dot{a}}{a}\Big) = p_g - p_0 + \frac{2\rho_{l0} R}{1+\chi}\,\frac{d^2 \Delta R(t - t_c)}{dt^2} + \frac{1-\chi}{1+\chi}\,\frac{d}{dt}\Big(\frac{a}{C_l}\,(p_g - p_0)\Big). \tag{3.3}$$

For $\chi = 0$, when the piston is as if "transparent" for waves diverging from the gas bubble, from (3.3) we obtain an equation which coincides in form with Eq. (2.13) for the initial stage of oscillations.

Figure 4 shows the effect of the oscillation frequency ΔR_A of the piston on the bubble dynamics during one period ($t_c \leqslant t \leqslant t_c + t^{(R)}$) [$0 \leqslant t' \leqslant t^{(R)}$ and $t' = t - t_c$]. The open points correspond to the maximum values of the effective parameter $\Delta\theta_{(m)}$ during one period. The calculations were carried out for the above-indicated values of the parameters of the fluid–gas bubble system. The frequency of oscillations of the piston was taken to be equal to $\omega^{(R)} = 2\pi \cdot 15$ kHz [$t^{(R)} = 67 \cdot 10^{-6}$ sec], which corresponds to the first resonance frequency for this liquid volume ($R = 5 \cdot 10^{-2}$ m). In the numerical solution (3.3), it was assumed that $\chi = 0.5$ and $a = a_0$ and $\dot{a} = 0$ ($t = t_c$). For the solution to be periodic (to correspond to a steady-state regime) under the given initial conditions, the condition $a = a_0$ and $\dot{a} = 0$ for $t = t_c + t^{(R)}$ should be satisfied. To do this, the sum of the period of the first leading oscillation of the bubble and the extension of the oscillating "tail" should decrease in one period of oscillations of the piston. It is noteworthy that by choosing the initial conditions in this way, in the general case we restrict the range of possible periodic solutions. In the absence of acoustic losses outside the fluid–bubble system ($\chi = 1$), damping of the oscillating "tail" requires a dissipative mechanism which is ensured in this case by introducing the effective viscosity; the value of this viscosity can exceed considerably the magnitude of the actual viscosity (in particular, owing to the interphase heat transfer). It is seen from Fig. 4 that an increase in the oscillation frequency of the piston causes an increase in the amplitude and period of the first leading oscillation of the bubble.

Figure 5 shows the bubble radius versus the time for various values of the coefficient χ, which takes into account the acoustic radiation to the ambient medium for $\Delta R_A = 0.24 \cdot 10^{-6}$ m.

The authors thank N. K. Vakhitova for assistance in computations.

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